INTEGRALS

(Expected Marks: 12)

Standard Integrals:

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
2. $\int \frac{1}{x} dx = \log|x| + c$
3. $\int e^x \, dx = e^x + c$
4. $\int a^x \, dx = \frac{a^x}{\log a} + c$
5. $\int \sin x dx = -\cos x + c$
6. $\int \cos x dx = \sin x + c$
7. $\int sec^2 x dx = \tan x + c$
8. $\int cosec^2 x dx = -\cot x + c$
9. $\int \sec x \tan x \, dx = \sec x + c$
10. $\int cosec\ x \ \cot x \ dx = -cosec\ x + c$
11. $\int \cot x \, dx = \log|\sin x| + c$
12. $\int \tan x \, dx = \log|\sec x| + c$
13. $\int \sec x \, dx = \log|\sec x + \tan x| + c$
14. $\int cosec\ x dx = \log|cosec\ x - \cot x| + c$
15. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + c$
16. $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1} \frac{x}{a} + c$
17. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$
18. $\int \frac{-1}{a^2+x^2} dx = \frac{1}{a} \cot^{-1} \frac{x}{a} + c$
19. $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$
20. $\int \frac{-1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \text{cosec}^{-1} \frac{x}{a} + c$

$$\boldsymbol{\int \frac{1}{x^n} dx = \frac{-1}{(n-1)x^{n-1}} + c}$$

$$\boldsymbol{\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c}$$

Tips to remember:

1. In any of the fundamental integration formulas given above if in place of 'x' we have 'ax+b' ,then the same formula is applicable but we must divide by coefficient of 'x' or derivative of 'ax+b'.
2. If the integrand is a rational algebraic function and if the degree of numerator is greater than or equal to the degree of denominator ,then always divide the numerator by denominator and use the result:

$$\frac{Nr}{Dr} = Quotient + \frac{Remainder}{Dr}$$

3. To evaluate integrals of the form $\int sin^m xdx$ or $\int cos^m xdx$ where $m \leq 4$, express $sin^m x$ and $cos^m x$ in terms of sines and cosines of multiples of x by using the following trigonometric identities:

$sin^2 x = \frac{1-\cos 2x}{2}$; $cos^2 x = \frac{1+\cos 2x}{2}$;

$sin^3 x = \frac{3sinx - \sin 3x}{4}$; $cos^3 x = \frac{\cos 3x + 3\cos x}{4}$

4. Some useful substitutions in evaluating integrals:

Expression	Substitution
$a^2 + x^2$	$x = \text{a} \tan\theta$ or $x = \text{a} \cot\theta$
$a^2 - x^2$	$x = a\sin\theta$ or $x = a\cos\theta$
$x^2 - a^2$	$x = a\sec\theta$ or $x = a\, cosec\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$

Some Special Integrals:

1. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
2. $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$
3. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$
4. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c$
5. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \log\left|x + \sqrt{a^2+x^2}\right| + c$
6. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2-a^2}\right| + c$

IMPORTANT TYPES OF INTEGRATION

1.Evaluation of integrals of the type $\int \frac{\mathbf{1}}{\boldsymbol{ax^2+bx+c}}\boldsymbol{dx}$

Step 1: Make the coefficient of x^2 unity.

Step 2: Add and subtract the square of the half of coefficient of 'x' to express

$$ax^2 + bx + c \text{ in the form: } a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

Step 3: Use the suitable formula from the following formulas:

1. $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
2. $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$
3. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

2.Evaluation of integrals of the type $\int \frac{\mathbf{1}}{\sqrt{\boldsymbol{ax^2+bx+c}}}\boldsymbol{dx}$

Step 1: Make the coefficient of x^2 unity.

Step 2:Add and subtract the square of the half of coefficient of 'x' inside the square root to express the quantity inside the square root in the form

$$\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} \text{ or } \frac{4ac-b^2}{4a^2} - \left(x+\frac{b}{2a}\right)^2.$$

Step 3:Use the suitable formula from the following formulas:

1. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c$
2. $\int \frac{1}{\sqrt{a^2+x^2}}dx = \log\left|x+\sqrt{a^2+x^2}\right| + c$
3. $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$

3.Evaluation of integrals of the type $\int \frac{\boldsymbol{px+q}}{\boldsymbol{ax^2+bx+c}}\boldsymbol{dx}$

Step 1:Write the numerator $px + q$ in the following form:

$$px + q = \lambda\left\{\frac{\boldsymbol{d}}{\boldsymbol{dx}}(\mathrm{ax}^2 + bx + c)\right\} + \boldsymbol{\mu}$$

$$\text{i.e. } px + q = \lambda(2ax + b) + \mu$$

Step 2:Obtain the values of λ and μ by equating the coefficient of like powers of 'x' on both sides.

Step 3:Replace $px+q$ by $\lambda(2ax+b)+\mu$ in the given integral to get

$$\int \frac{px+q}{ax^2+bx+c}dx = \lambda\int \frac{2ax+b}{ax^2+bx+c}dx + \mu\int \frac{1}{(ax^2+bx+c)}dx \text{ and integrate.}$$

4 .Evaluation of integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Step 1:Write the numerator $px+q$ in the following form:

$$px+q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu$$

$$\text{i.e. } px+q=\lambda(2ax+b)+\mu$$

Step 2:Obtain the values of λ and μ by equating the coefficient of like powers of 'x' on both sides.

Step 3:Replace $px+q$ by $\lambda(2ax+b)+\mu$ in the given integral to get

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = \lambda\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + \mu\int \frac{1}{\sqrt{ax^2+bx+c}}dx \text{ and integrate.}$$

5 .Integrals of the form

$$\int \frac{1}{a\sin x+b\cos x}dx, \int \frac{1}{a+b\sin x}dx, \int \frac{1}{a+b\cos x}dx, \int \frac{1}{a\sin x+b\cos x+c}dx$$

Step 1:Put $\sin x = \frac{2\tan\frac{x}{2}}{1+tan^2\frac{x}{2}}$, $\cos x = \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}$

Step 2: Replace $1+tan^2\frac{x}{2}$ in the numerator by $sec^2\frac{x}{2}$

Step 3: Put $tan\frac{x}{2}=t$ so that $\frac{1}{2}sec^2\frac{x}{2}dx=dt$

This substitution reduces the integral in the form $\int \frac{1}{at^2+bt+c}dt$.

6 .Integration by Partial Fractions:

If the Integrand $\frac{P(x)}{Q(x)}$ is a proper rational function ,we can write the integrand as a sum of simpler rational functions by partial fraction decomposition.

7 .Integration by Parts

If $f(x)$ and $g(x)$ are two functions then,

$$\int f(x)\, g(x) dx = f(x) \int g(x)\, dx - \int \left[f'(x) \int g(x)\, dx \right] dx$$

i.e. **"The integral of the product of two functions = (first function) X (integral of the second function) –Integral of {(derivative of the first function) X (integral of the second function)}"**

> *Note*:
> We can choose the first function as the function which comes first in the word ILATE, where
> I –stands for the inverse trigonometric function
> L-stands for the logarithmic functions
> A –stands for the algebraic functions
> T –stands for trigonometric functions
> E –stands for exponential functions

8 .Integrals of the form $\int \boldsymbol{e^x \{f(x) + f'(x)\} dx}$

$$\int e^x\{\boldsymbol{f(x) + f'(x)}\}\, dx = e^x f(x) + C$$

9 .Some important integrals

1. $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$
2. $\int \sqrt{a^2 + x^2} dx = \frac{1}{2} x\sqrt{a^2 + x^2} + \frac{1}{2} a^2 log\left|x + \sqrt{a^2 + x^2}\right| + c$
3. $\int \sqrt{x^2 - a^2} dx = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{1}{2} a^2 log\left|x + \sqrt{x^2 - a^2}\right| + c$

10 .Integrals of the form $\int \boldsymbol{(px + q)\sqrt{ax^2 + bx + c}\, dx}$

Step 1: Express $px + q$ as

$$px + q = \lambda \left\{\frac{d}{dx}\left(\mathrm{ax}^2 + bx + c\right)\right\} + \mu$$

$$\text{i.e. } px + q = \lambda(2ax + b) + \mu$$

Step 2:Obtain the values of λ and μ by equating the coefficient of like powers of 'x' on both sides.

Step 3: Replace $px + q$ by $\lambda(2ax + b) + \mu$ in the integral to obtain

$$\int (px+q)\sqrt{ax^2+bx+\ c}\,dx$$
$$= \lambda \int (2ax+b)\sqrt{ax^2+bx+\ c}\,dx \ + \ \mu \int \sqrt{ax^2+bx+\ c}\,dx$$

11 .Integrals of the form $\int \frac{x^2+1}{x^4+\lambda x^2+1}dx$, $\int \frac{x^2-1}{x^4+\lambda x^2+1}dx$,$\int \frac{1}{x^4+\lambda x^2+1}dx$, *where* $\lambda \in R$

Step 1: Divide numerator and denominator by x^2

Step 2: Express the denominator of the integrand in the form $\left(x+\frac{1}{x}\right)^2 \pm k^2$

Step 3: Introduce $d\left(x+\frac{1}{x}\right)$ or $d\left(x-\frac{1}{x}\right)$ or both in the numerator.

Step 4: Substitute $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$ as the case may be.

This substitution reduces the integral in one of the following forms $\int \frac{1}{x^2+a^2}dx, \int \frac{1}{x^2-a^2}dx.$

12. Integrals of the form $\int e^{ax}\sin bx\,dx$, $\int e^{ax}\cos bx\,dx$.

$I = \int e^{ax}\sin bx\,dx$,

$$\Rightarrow I = e^{ax}\int \sin bx\,dx - \int\left(\frac{d}{dx}(e^{ax})\int \sin bx\,dx\right)dx,$$

$$\Rightarrow I = -e^{ax}\frac{\cos bx}{b} - \int ae^{ax}\left(-\frac{\cos bx}{b}\right)dx,$$

$$\Rightarrow I = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx,$$

$$\Rightarrow I = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}dx\right],$$

$$\Rightarrow I = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx\ ,$$

$$\Rightarrow I = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I,$$

$$\Rightarrow I + I.\frac{a^2}{b^2} = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx),$$

$$\Rightarrow\ I\left(\frac{a^2+b^2}{b^2}\right) = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx),$$

$$\therefore \boldsymbol{I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}.$$

13. Integrals of the form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$

Step 1. Write

$$Numerator = \lambda \frac{d}{dx}(Denominator) + \mu(Denominator)$$

$$ie.\ a\sin x + b\cos x = \lambda(c\cos x - d\sin x) + \mu(c\sin x + d\cos x)$$

Step 2. Obtain the values of λ and μ by equating the coefficients of $\sin x$ and $\cos x$ on both sides.

Step 3. Replace numerator in the integrand by

$\lambda(c\cos x - d\sin x) + \mu(c\sin x + d\cos x)$ to obtain

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx = \lambda\int \frac{c\cos x - d\sin x}{c\sin x + d\cos x}dx + \mu\int \frac{c\sin x + d\cos x}{c\sin x + d\cos x}dx$$

PRACTICE QUESTIONS

a) Very short answer questions (1 mark each)

Evaluating the following.

1. $\int \frac{2\cos x}{3\sin^2 x}dx$
2. $\int \frac{\sin 4x}{\cos 2x}dx$
3. $\int e^{3\log x}(x^4+1)^{-1}dx$
4. $\int sec^3 x\tan x\,dx$
5. $\int x^3\sin x^4\,dx$
6. $\int \frac{cos^3 x}{\sqrt{\sin x}}dx$
7. $\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$
8. $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$
9. $\int e^x(\tan x + \log\sec x)dx$
10. $\int(\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x})dx$
11. $\int \frac{1}{1-\sin x}dx$
12. $\int \frac{x}{4+x^4}dx$
13. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$
14. $\int \frac{\cos 2x - 1}{\cos 2x + 1}dx$
15. $\int e^x(1 - \cot x + cot^2 x)dx$

Answers

1. $\frac{-2}{3}cosec\,x + c$, **2.** $-\cot 2x + c$, **3.** $\frac{1}{4}\log(x^4+1)+c$, **4.** $\frac{sec^3 x}{3} + c$,

5. $\frac{-1}{4}\cos x^4 + c$, **6.** $2\sqrt{\sin x} - \frac{2}{5}(\sin x)^{5/2} + c$, **7.** $-cos(\tan^{-1}x) + c$,

8. $e^x\frac{1}{x} + c$, **9.** $e^x\log\sec x + c$, **10.** $\frac{\pi}{2}x + c$, **11.** $\tan x + \sec x + c$,

12. $\frac{1}{4}\tan^{-1}\frac{x^2}{2} + c$, **13.** $2\sin\sqrt{x} + c$, **14.** $x - \tan x + c$, **15.** $-e^x\cot x + c$

b) Short answer type questions(4 marks each)

1. $\int \boldsymbol{sin^4 x\, dx}$

$$\int sin^4 x\, dx = \int (sin^2 x)^2\, dx,$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx,$$

$$= \int \frac{1-2\cos 2x + cos^2\, 2x}{4}\, dx,$$

$$= \int \frac{1-2\cos 2x + \left(\frac{1+\cos 4x}{2}\right)}{4}\, dx,$$

$$= \frac{1}{8}\int (2 - 4\cos 2x + 1 + \cos 4x)\, dx,$$

$$= \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x) dx,$$

$$= \boldsymbol{\frac{1}{8}\left[3x - 2\sin 2x + \frac{\sin 4x}{4}\right] + c}.$$

2. $\int \boldsymbol{\frac{1}{\sin(x-a)\cos(x-b)}\, dx}$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}\, dx,$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}\, dx,$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(x-a)\cos(x-b)+\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)},$$

$$= \frac{1}{\cos(a-b)}\int (\cot(x-a) + \tan(x-b))\, dx,$$

$$= \frac{1}{\cos(a-b)}[\log|\sin(x-a)| - \log|cos(x-b)|] + c,$$

$$= \boldsymbol{\frac{1}{\cos(a-b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + c}.$$

3. $\int \boldsymbol{\frac{1}{x(x^5+1)}\, dx}$

$$= \int \frac{x^4}{x^5(x^5+1)}\, dx, \qquad put\ \ x^5 = t\,, x^4 dx = \frac{1}{5}dt,$$

$$= \frac{1}{5}\int \frac{1}{t^2+t}\, dt,$$

$$= \frac{1}{5}\int \frac{1}{t^2+t+\frac{1}{4}-\frac{1}{4}}\, dt,$$

$$= \frac{1}{5}\int \frac{1}{\left(t+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dt,$$

$$= \frac{1}{5}\log\left|\frac{t+\frac{1}{2}-\frac{1}{2}}{t+\frac{1}{2}+\frac{1}{2}}\right| + c,$$

$$= \frac{1}{5}\log\left|\frac{t}{t+1}\right| + c,$$

$$= \boldsymbol{\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right| + c}.$$

4. $\int \boldsymbol{\frac{\cos(x+a)}{\sin(x+b)}\,dx}$

$$I = \int \frac{\cos(x+a)}{\sin(x+b)}\,dx \qquad \text{Put } x+b = t \text{ } and \text{ } dx = dt$$

$$= \int \frac{\cos(t-b+a)}{\sin(t)}\,dt,$$

$$= \int \frac{\cos\{t+(a-b)\}}{\sin(t)}\,dt,$$

$$= \int \frac{\cos t\cos(a-b)-\sin t\sin(a-b)}{\sin t}\,dt,$$

$$= \int [\cot t\cos(a-b)-\sin(a-b)]dt,$$

$$= \cos(a-b)\int \cot t\,dt - \sin(a-b)\int dt,$$

$$= \cos(a-b)\log|\sin t| - \sin(a-b)t + c$$

$$= \boldsymbol{\cos(a-b)\log|\sin(x+b)| - (x+b)\sin(a-b) + c}.$$

5. $\int \boldsymbol{\frac{1}{1+\cot x}\,dx}$

$$I = \int \frac{1}{1+\cot x}\,dx \quad \Rightarrow I = \int \frac{1}{1+\frac{\cos x}{\sin x}}\,dx,$$

$$\Rightarrow I = \int \frac{\sin x}{\sin x+\cos x}\,dx,$$

$$Let \sin x = \lambda\frac{d}{dx}(\sin x+\cos x) + \mu(\sin x+\cos x)$$

$$\Rightarrow \sin x = \lambda(\cos x - \sin x) + \mu(sinx + \cos x)$$

Now by comparing coefficients of $\sin x$ and $\cos x$ on both sides

we get $0 = \lambda + \mu \; and \; 1 = -\lambda + \mu$

$$\Rightarrow \lambda = \frac{-1}{2}, \mu = \frac{1}{2}$$

$\Rightarrow I = \int \frac{\lambda(\cos x - \sin x) + \mu(sinx + \cos x)}{sinx + \cos x}$,

$\Rightarrow I = \lambda \int \frac{(\cos x - \sin x)}{sinx + \cos x} dx + \mu \int \frac{(sinx + \cos x)}{(sinx + \cos x)} dx,$

$\Rightarrow I = \lambda \int \frac{dt}{t} + \mu \int dx \,, \quad where \; sinx + \cos x = t \,,$

$\Rightarrow I = \lambda \log|t| + \mu x + c,$

$$\Rightarrow \boldsymbol{I = \frac{-1}{2} \log|sinx + \cos x| + \frac{1}{2} x + c}$$

6. $\int \frac{\mathbf{1}}{\sqrt{\mathbf{9+8x-x^2}}} \boldsymbol{dx}$

$$= \int \frac{1}{\sqrt{-(x^2-8x+9)}} dx,$$

$$= \int \frac{1}{\sqrt{-(x^2-8x+16-16-9)}} dx,$$

$$= \int \frac{1}{\sqrt{-(x^2-8x+16-25)}} dx,$$

$$= \int \frac{1}{\sqrt{-\{(x-4)^2-5^2\}}} dx,$$

$$= \int \frac{1}{\sqrt{5^2-(x-4)^2}} dx = \boldsymbol{\sin^{-1}\left(\frac{x-4}{5}\right) + c}$$

✓ ***Alternate method to evaluate Integrals of the form***

$\int \frac{\boldsymbol{dx}}{\sqrt{(\boldsymbol{ax+b})(\boldsymbol{cx+d})}}$

Step 1: Take $(cx + d) = t^2$ and $x = \frac{t^2-d}{c}$,$dx = \frac{1}{c} 2tdt$

Step 2:Then the integral $\int \frac{\boldsymbol{dx}}{\sqrt{(\boldsymbol{ax+b})(\boldsymbol{cx+d})}}$ changes to

$$\frac{1}{c} \int \frac{2tdt}{\sqrt{\left[a\left(\frac{t^2-d}{c}\right)+b\right]t^2}}$$

$$ie.\frac{2}{c}\int \frac{dt}{\sqrt{\left[a\left(\frac{t^2-d}{c}\right)+b\right]}}$$

Example: $\int \frac{\boldsymbol{dx}}{\sqrt{\boldsymbol{x^2+3x+2}}}$

$$I = \int \frac{\boldsymbol{dx}}{\sqrt{\boldsymbol{x^2+3x+2}}}$$

$\Rightarrow I = \int \frac{dx}{\sqrt{(x+2)(x+1)}}$,

put $x+1 = t^2, x = t^2 - 1$ $and\ dx = 2td,$

$\Rightarrow I = \int \frac{2tdt}{\sqrt{(t^2+1)t^2}}$,

$\Rightarrow I = 2\int \frac{dt}{\sqrt{t^2+1}}$

$\Rightarrow I = 2\log\left|t+\sqrt{t^2+1}\right|$+c

$\Rightarrow I = 2\log\left|\sqrt{x+1}+\sqrt{x+2}\right|$+c.

Try yourself:

Evaluate

1. $\int \frac{dx}{\sqrt{x^2-5x+6}}$
2. $\int \frac{dx}{\sqrt{2x^2+3x-2}}$

- ***A special integral of the form*** $\int \frac{\boldsymbol{dx}}{(\boldsymbol{ax+b})\sqrt{\boldsymbol{cx+d}}}$ ***(optional)***

Step 1: Take $(cx+d) = t^2$ and $= \frac{t^2-d}{c}$,$dx = \frac{1}{c}2tdt$

Step 2:Then the integral $\int \frac{\boldsymbol{dx}}{(\boldsymbol{ax+b})\sqrt{\boldsymbol{cx+d}}}$ changes to

$$\frac{1}{c}\int \frac{2tdt}{\left[a\left(\frac{t^2-d}{c}\right)+b\right]\sqrt{t^2}} = \frac{2}{c}\int \frac{dt}{\left[a\left(\frac{t^2-d}{c}\right)+b\right]}$$

Example $\int \frac{\boldsymbol{dx}}{\boldsymbol{(2x+3)\sqrt{x+5}}}$ **(HOTS)**

$I = \int \frac{dx}{(2x+3)\sqrt{x+5}}$ $\quad let\ x + 5 = t^2, \Rightarrow x = t^2 - 5\ and$

$dx = 2tdt$

$\Rightarrow I = \int \frac{2tdt}{[2(t^2-5)+3]\sqrt{t^2}}$

$$\Rightarrow I = \int \frac{2dt}{2t^2 - 7}$$

$$\Rightarrow I = \int \frac{dt}{t^2 - \frac{7}{2}}$$

$$\Rightarrow I = \int \frac{dt}{t^2 - \left(\sqrt{7/2}\right)^2}$$

$$\Rightarrow I = \frac{1}{2\sqrt{7/2}} \log \left| \frac{t - \sqrt{7/2}}{t + \sqrt{7/2}} \right|$$

$$\Rightarrow \boldsymbol{I} = \frac{\mathbf{1}}{\sqrt{\mathbf{14}}} \mathbf{log} \left| \frac{\sqrt{\boldsymbol{x}+\mathbf{5}} - \sqrt{7/2}}{\sqrt{\boldsymbol{x}+\mathbf{5}} + \sqrt{7/2}} \right| + \boldsymbol{c}$$

Q. Evaluate:

$\int \{\mathbf{sin}(\mathbf{log}\,\boldsymbol{x}) + \mathbf{cos}\,(\mathbf{log}\,\boldsymbol{x})\}\boldsymbol{dx}$.

$$I = \int \{\mathbf{sin}(\mathbf{log}\,\boldsymbol{x}) + \mathbf{cos}(\,\mathbf{log}\,\boldsymbol{x})\}\boldsymbol{dx}$$

Let $\log x = t, then\ x = e^t \Rightarrow dx = e^t dt$

$$I = \int e^t(\sin t + \cos t)\,dt$$

$$= e^t \sin t + c$$

$$= e^{\log x} \sin(\log x) + c$$

$$= x \sin(\log x) + c$$

Evaluate the following

1. $\int cos^4 x\, dx$

2. $\int \cos 2x . \cos 4x\, dx$

{Hint: Multiply Nr & Dr by 2}

3. $\int \frac{\sin 2x}{a^2 sin^2 x + b^2 cos^2 x} dx$

4. $\int \frac{\sin x}{sin(x-a)} dx$

$[Hint: Put\ x - a = t]$

5. $\int \frac{1}{e^x+1} dx$

$[Hint: Divide\ Nr\ \&\ Dr\ by e^x]$

6. $\int \frac{\sin 2x}{(a+b\cos x)^2} dx$

7. $\int \frac{(x^4-x)^{1/4}}{x^5} dx$

8. $\int \frac{1}{x^2-4x+8} dx$

9. $\int \frac{x}{x^4+x^2+1} dx$

10. $\int \frac{2x}{\sqrt{1-x^2-x^4}} dx$

11. $\int \frac{x+2}{2x^2+6x+5} dx$

12. $\int \frac{1}{1-\tan x} dx$

13. $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

14. $\int \sqrt{\frac{1-x}{1+x}} dx$

15. $\int tan^4 x\, dx$

16. $\int \sqrt{\tan x}\, dx$ $[Hint: \tan x = t^2]$

17. $\int \frac{1}{x^4-5x^2+16} dx$

$\left[\begin{matrix} Hint: Multiply\ and\ divide\ by\ 8, then \\ divide\ Nr\ and\ Dr\ by x^2 \end{matrix}\right]$

18. $\int \frac{dx}{3+2\sin x+\cos x}$

19. $\int \frac{2x+3}{\sqrt{x^2+x+1}} dx$

20. $\int \frac{dx}{\sqrt{16-2x-2x^2}}$

21. $\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

$[Hint: Put\ \log x = t, x = e^t]$

22. $\int e^x \left(\frac{x^2+1}{(x+1)^2}\right) dx$

23. $\int \frac{\sin^{-1} x}{x^2} dx$

24. $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

25. $\int sin(\log x)\, dx$

$[Hint: \log x = t\, , x = e^t]$

26. $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$

27. $\int (\sin^{-1} x)^2 dx$

28. $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$

29. $\int e^{2x} \left(\frac{1+\sin 2x}{1+\cos 2x}\right) dx$

30. $\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$

31. $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$

32. $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$

$[Hint: Put x = atan^2\theta]$

33. $\int \frac{\sqrt{x^2+a^2}}{x} dx$

$[Hint: x = a\tan\theta]$

34. $\int \frac{sin\, x}{sin\, 4x} dx$

Answers

1. $\frac{1}{8}\left(3x + 2\sin 2x + \frac{\sin 4x}{4}\right) + c$ **2.** $\frac{1}{2}\left(\frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right) + c$

3. $\frac{1}{a^2-b^2}\log|a^2 sin^2 x + b^2 cos^2 x| + c$

4. $\sin a \log|\sin(x-a)| + (x-a)\cos a + c$ **5.** $-\log|1 + e^{-x}| + c$

6. $\frac{-2}{b^2}\left[\log|a + b\cos x| + \frac{a}{a+b\cos x}\right] + c$ **7.** $\frac{4}{15}\left(1 - \frac{1}{x^3}\right)^{5/4} + c$

8. $\frac{1}{2}\tan^{-1}\frac{x-2}{2} + c$ **9.** $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$ **10.** $\sin^{-1}\frac{2x^2+1}{\sqrt{5}} + c$

11. $\frac{1}{4}\log(2x^2 + 6x + 5) + \frac{1}{2}\tan^{-1}(2x+3) + c$

12. $\frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + c$ **13.** $2\sqrt{\tan x} + c$

14. $\sin^{-1} x + \sqrt{1-x^2} + c$ **15.** $\frac{1}{3}tan^3 x - \tan x + x + c$

16. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right| + c$

17. $\frac{1}{8\sqrt{3}}\tan^{-1}\frac{x^2-4}{\sqrt{3}x} - \frac{1}{16\sqrt{13}}\log\left|\frac{x^2-\sqrt{13}x+4}{x^2+\sqrt{13}x+4}\right| + c$ **18.** $\tan^{-1}\left(\tan\frac{x}{2} + 1\right) + c$

19. $2\sqrt{x^2+x+1} + 2\log\left|x + \frac{1}{2} + \sqrt{x^2+x+1}\right| + c$

20. $\frac{1}{\sqrt{2}}\sin^{-1}\frac{2x+1}{\sqrt{33}} + c$ **22.** $x\log(\log x) - \frac{x}{\log x} + c$

23. $-\frac{\sin^{-1} x}{x} + \log\left|\frac{1-\sqrt{1-x^2}}{x}\right| + c$ **24.** $\sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + c$

25. $\frac{x}{2}\{\sin(\log x) - \cos(\log x)\} + c$ **26.** $\sin^{-1}\left(\frac{e^x+2}{3}\right) + c$

27. $x(\sin^{-1} x)^2 - 2\left[-\sqrt{1-x^2}\sin^{-1} x + x\right] + c$

28. $\frac{2}{\pi}\left[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}\right] - x + c$ **29.** $\frac{1}{2}e^{2x}\tan x + c$

30. $\frac{x}{\log x} + c$ **31.** $x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + c$

32. $a\left[\left(\frac{x+a}{a}\right)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}}\right] + c$ **33.** $\sqrt{x^2+a^2} - \frac{a}{2}\log\frac{\sqrt{x^2+a^2}+a}{\sqrt{x^2+a^2}-a} + c$

34. $-\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + c$

DEFINITE INTEGRALS

Definite integrals as the limit of a sum:

Let $f(x)$be a continuous real valued function defined on [a,b], then

$\int_a^b \boldsymbol{f(x)\,dx} = \mathbf{lim}_{n\to\infty} \boldsymbol{h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)]},$

$$where\ h = \frac{b-a}{n}$$

Or

$\int_a^b \boldsymbol{f(x)\,dx} = \mathbf{lim}_{h\to 0} \boldsymbol{h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)]},$

$$[\because n \to \infty \Leftrightarrow h \to 0]$$

Or

$$\int_a^b \boldsymbol{f(x)\,dx} = \lim_{h\to 0} \boldsymbol{h} \sum_{i=0}^{n-1} \boldsymbol{f(a+ih)}$$

The following results will be useful in evaluating definite integral as limit of sums:

$$1 + 2 + 3 + \cdots + (n-1) = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 = \sum_{i=0}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 = \sum_{i=0}^{n-1} i^3 = \left\{\frac{n(n-1)}{2}\right\}^2$$

$$a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{i=0}^{n-1} ar^i = a\left(\frac{r^n - 1}{r-1}\right), r \neq 1$$

Example

1. Evaluate $\int_1^3 (2x^2 + 5)dx$ as limit of sums:

   Here ,$a = 1, b = 5, f(x) = (2x^2 + 5) and\ h = \frac{3-1}{n} \Rightarrow \boldsymbol{nh = 2}$

$\therefore \int_1^3 (2x^2 + 5)dx$

$$= \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \cdots + f(1+(n-1)h)]$$

$$= \lim_{h\to 0} h\{[2(1)^2+5] + [2(1+h)^2+5] + [2(1+2h)^2+5] + \cdots$$

$$+ [2(1+(n-1)h)^2+5]\}$$

$$= \lim_{h\to 0} h\{2[1^2 + (1+h)^2 + (1+2h)^2 + \cdots + (1+(n-1)h)^2] + 5n\}$$

$$= \lim_{h\to 0} h\{2[n + 2h(1+2+3+\cdots+(n-1)) + h^2(1^2+2^2+3^2+\cdots+(n-1)^2)] + 5n\}$$

$$= \lim_{h\to 0} h\left[2\left\{n + 2h\frac{n(n-1)}{2} + h^2\frac{n(n-1)(2n-1)}{6}\right\} + 5n\right]$$

$$= \lim_{h\to 0} h\left\{2n + 2hn(n-1) + h^2\frac{n(n-1)(2n-1)}{3} + 5n\right\}$$

$$= \lim_{h\to 0}\left\{7nh + 2h^2n^2\left(1-\frac{1}{n}\right) + \frac{1}{3}h^3n^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right\}$$

$$= \lim_{h\to 0}\left\{7.2 + 2(2)^2\left(1-\frac{1}{n}\right) + \frac{1}{3}(2)^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right\}$$

Now as $h \to 0, n \to \infty$

$\therefore \int_1^3 (2x^2+5)dx = \lim_{n\to\infty}\left\{7.2 + 2(2)^2\left(1-\frac{1}{n}\right) + \frac{1}{3}(2)^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right\}$

$$= 14 + 8(1-0) + \frac{8}{3}(1-0)(2-0)$$

$$= 14 + 8 + \frac{16}{3} = \mathbf{\frac{82}{3}}$$

<u>Alternate method</u>

$\int_1^3 (2x^2+5)dx,$

Here ,$a = 1, b = 5, f(x) = (2x^2+5) and\ h = \frac{3-1}{n} \Rightarrow \boldsymbol{nh = 2,}$

$x = a + ih = 1 + ih$

$$\int_1^3 (2x^2+5)dx = \lim_{h\to 0} h\sum_{i=0}^{n-1} f(1+ih)$$

$$= \lim_{h\to 0} h\sum_{i=0}^{n-1}\{2(1+ih)^2+5\}$$

$$= \lim_{h\to 0} h\sum_{i=0}^{n-1}\left\{2\left(1+2ih+i^2h^2\right)+5\right\}$$

$$= \lim_{h\to 0} h \sum_{i=0}^{n-1} \{7 + 4ih + 2i^2h^2\}$$

$$= \lim_{h\to 0} h \left\{ \sum_{i=0}^{n-1} 7 + 4h \sum_{i=0}^{n-1} i + 2h^2 \sum_{i=0}^{n-1} i^2 \right\}$$

$$= \lim_{h\to 0} h \left\{ 7n + 4h \frac{n(n-1)}{2} + 2h^2 \frac{n(n-1)(2n-1)}{6} \right\}$$

$$= \lim_{h\to 0} \left\{ 7nh + 2h^2n^2 \left(1 - \frac{1}{n}\right) + \frac{1}{3} h^3 n^3 \left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) \right\}$$

$$= \lim_{h\to 0} \left\{ 7.2 + 2(2)^2 \left(1 - \frac{1}{n}\right) + \frac{1}{3} (2)^3 \left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) \right\}$$

Now as $h \to 0, n \to \infty$

$\therefore \int_1^3 (2x^2 + 5)dx = \lim_{n\to\infty} \left\{ 7.2 + 2(2)^2 \left(1 - \frac{1}{n}\right) + \frac{1}{3} (2)^3 \left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) \right\}$

$$= 14 + 8(1-0) + \frac{8}{3}(1-0)(2-0)$$

$$= 14 + 8 + \frac{16}{3} = \mathbf{\frac{82}{3}}$$

<u>Practice Questions:</u>

1. $\int_1^3 (x^2 + x)dx$
2. $\int_0^3 (2x^2 + 3x + 5)\, dx$
3. $\int_0^2 (x^2 + x)dx$
4. $\int_0^1 e^{2-3x}\, dx$
5. $\int_{-1}^1 e^x dx$
6. $\int_0^4 (x + e^{2x})\, dx$
7. $\int_2^4 2^x dx$

Fundamental theorem of Integral Calculus

Let $f(x)$ be continuous function defined on the closed interval $[a, b]$ and $F(x)$ be an anti derivative of $f(x)$, then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

- **Evaluation of Definite Integrals by Substitution**

  When the variable in a definite integral is changed, the substitution in terms of new variable should be effected at three places.

  a) In the integrand
  b) In the differential, say, dx
  c) In the limits.

Properties of Definite Integrals

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$
2. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$,where $a < c < b$
4. If $f(x)$ is a continuous function defined on $[a, b]$,then
$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$
5. If $f(x)$ is a continuous function defined on $[0, a]$,then
$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$
6. If $f(x)$ is continuous function defined on $[-a, a]$, then
$$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx\ , if\ f(x) is\ an\ even\ function \\ 0 \qquad\qquad , if\ f(x) is\ an\ odd\ function \end{cases}$$

{A function $f(x)$ is called an <u>even</u> function if $f(-x) = f(x)$ and on <u>odd</u> function if $f(-x) = -f(x)$}

7. If $f(x)$ is a continuous function defined on $[0,2a]$,then

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if\ f(2a-x) = f(x) \\ 0, & if\ f(2a-x) = -f(x) \end{cases}$$

<u>Problems on Removal of 'x'</u>

Let $I = \int_0^a xf(x)dx$ where $f(x)$a function of is x whose integral is known and $f(a-x) = f(x)$. Then,

$I = \int_0^a xf(x)dx,$

$\Rightarrow I = \int_0^a (a-x)f(a-x)dx$

$\Rightarrow I = a\int_0^a f(x)dx - \int_0^a xf(x)dx$

$\Rightarrow I = a\int_0^a f(x)dx\ - I$

$\Rightarrow 2I = a\int_0^a f(x)\,dx$

$\Rightarrow \boldsymbol{I = \frac{a}{2}\int_0^a f(x)\,dx}$**.**

<u>Practice Questions:</u>

<u>Very short answer questions:</u>

1. $\int_0^1 \frac{2x}{1+x^2}dx$
2. $\int_0^3 [x]\,dx$
3. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^7 x\, d\,x$
4. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$
5. $\int_1^{\sqrt{3}} \frac{1}{1+x^2}dx$

<u>Answers:</u> **1.**log 2 **2.** 3 **3.** 0 **4.** $\frac{\pi}{4}$ **5.** $\frac{\pi}{12}$

Short Answer questions:

Evaluate the following

1. $\int_0^{\frac{\pi}{4}} \sqrt{1-\sin 2x}\, dx$
2. If $\int_a^b x^3\, dx = 0$ and if $\int_a^b x^2\, dx = \frac{2}{3}$, find a and b.
3. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2x \log \sin x\, dx$
4. $\int_1^2 \left(\frac{x-1}{x^2}\right) e^x dx$
5. $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)} d\theta$
6. $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$
7. $\int_0^{\frac{\pi}{2}} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$
8. $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{sin^4 x + \cos^4 x} dx$
9. Find $\int_1^4 f(x)dx$ if $f(x) = |x-1| + |x-2| + |x-3|$
10. $\int_1^3 \frac{\sqrt{x}}{\sqrt{4-x}+\sqrt{x}} dx$
11. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}} dx$
12. $\int_0^{\frac{\pi}{2}} \log \tan x\, dx$

Answers

1 .$\sqrt{2}-1$	**2** .$a=-1, b=1$	**3** .$\frac{1}{4}\log 2 - \frac{\pi}{8} + \frac{1}{4}$	**4** .$e^2 - e$	
5 .$\log\frac{4}{3}$	**6** .$\frac{\pi}{4} - \frac{1}{2}\log 2$	**7** .$\sqrt{2}\pi$	**8** .$\frac{\pi}{2}$	**9** .$\frac{19}{2}$
10 .2	**11** .$\frac{\pi}{12}$	**12** .0		

Long Answer Questions:

1. $\int_0^{\pi} \frac{1}{5+4\cos x} dx$
2. $\int_0^{\frac{\pi}{4}} tan^3\, xdx$
3. $\int_0^{\frac{\pi}{4}} \log(1+\tan x) dx$
4. $\int_0^{\frac{\pi}{2}} \frac{sin^2 x}{\sin x + \cos x} dx$
5. $\int_0^{\pi} \frac{x}{1+\sin x} dx$
6. $\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{sin^4 x + cos^4 x} dx$
7. $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$
8. $\int_0^{\frac{\pi}{2}} \log \sin x\, dx$
9. $\int_0^{\pi} \frac{x \tan x}{\sec x . cosec\, x} dx$
10. $\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$
11. $\int_0^{\pi} \frac{x}{a^2 cos^2 x + b^2 sin^2 x} dx$
12. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin|x| + \cos|x|)\, dx$
13. $\int_0^{2\pi} |\sin x|\, dx$

Answers:

1. $\frac{\pi}{3}$,	**2.** $\frac{1}{2}(1-\log 2)$,	**3.** $\frac{\pi}{8}\log 2$,	**4.** $\frac{-1}{\sqrt{2}}\log(\sqrt{2}-1)$,	**5.** π ,
6. $\frac{\pi^2}{16}$,	**7.** $a\pi$,	**8.** $\frac{-\pi}{2}\log 2$,	**9.** $\frac{\pi^2}{4}$,	**10.** $a\left(\frac{\pi}{2}-1\right)$ **11.** $\frac{\pi^2}{2ab}$
12. 4	**13.** 4			

AREAS OF BOUNDED REGIONS

(Expected marks:6)

- Let $f(x)$ be a continuous function defined on $[a,b]$. Then, the area bounded by the curve $y=f(x)$, the $x-axis$ and the ordinates $x=a$ and $x=b$ is given by

$$\int_a^b y\,dx \quad \textbf{or} \quad \int_a^b f(x)dx$$

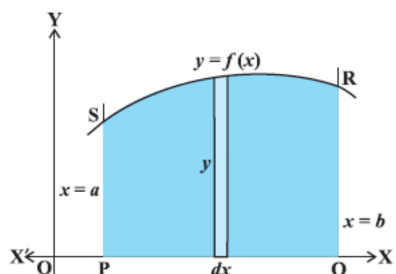
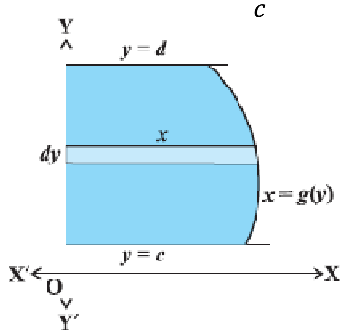


Fig 8.1

Note:

1. The area bounded by the curve $x=g(y)$, the $y-axis$ and the abscissae $y=c\ and\ y=d$ is given by

$$\int_c^d g(y)dy \quad or \quad \int_c^d x\,dy$$



2. If the curve $y = f(x)$ lies below $x - axis$, then the area bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ is given by

$$\left|\int_a^b y\, dx\right| \text{ or } \left|\int_a^b f(x)dx\right|$$

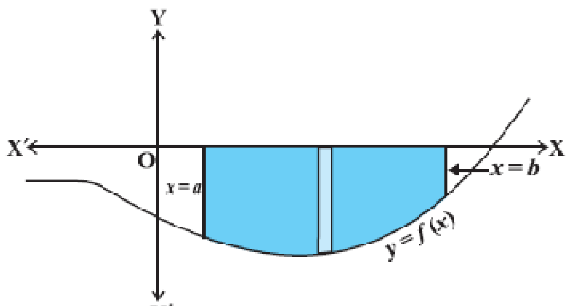
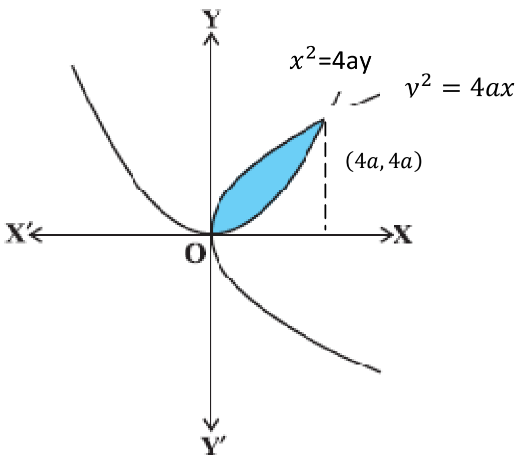


Fig 8.3

Example:

1. Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.



The equations of the given curves are $y^2 = 4ax$ and, $x^2 = 4ay$
Clearly both the equations represent parabolas in standard forms.
Solving $y^2 = 4ax$ and, $x^2 = 4ay$ we get the point of intersections as (0,0) and $(4a, 4a)$

Required area is= Area of the shaded region.
= (Area bounded by the parabola $y^2 = 4ax$ and $x - axis$ from 0 to 4a)−(Area bounded by the parabola $x^2 = 4ay$ and $x - axis$ from 0 to 4a)

$$= \int_0^{4a} \sqrt{4ax}dx - \int_0^{4a} \frac{x^2}{4a}dx$$

$$= \left[\frac{4\sqrt{a}}{3}x^{\frac{3}{2}} - \frac{x^3}{12a}\right]_0^{4a} = \frac{\mathbf{16a^2}}{\mathbf{3}}\text{sq.unit}$$

Practice Questions

1. Using integration, find the area bounded between the line $x = 4$ and the parabola $y^2 = 16x$.
2. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.
3. Find the area of the region bounded by the curves $y = x^2 + 2, y = x,$ $x = 0$ and $x = 3$.
4. Find the area bounded by the curves $y = x$ and $y = x^3$.
5. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $\frac{x}{a} + \frac{y}{b} = 1$.
6. Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.
7. Sketch the region common to the circle $x^2 + y^2 = 16$ and the parabola $x^2 = 6y$ and find its area using integration.
8. Find the area bounded by the lines $y = 4x + 5, y = 5 - x$ and $4y = x + 5$.
9. Using integration, find the area of the triangle ABC vertices have coordinates $A(2,5), B(4,7)$ and $C(6,2)$.

<u>HOTS</u>

10. Find the area of the region $\{(x, y): x^2 + y^2 \leq 1 \leq x + y\}$.
11. Draw a rough sketch of the region $\{(x, y): y^2 \leq 3x, 3x^2 + 3y^2 \leq 16\}$ and find the area enclosed by the region using method of integration.
12. Find the area of the circle $x^2 + y^2 = 16$ which is exterior to the parabola$y^2 = 6x$
13. Sketch the graph of $y = |x + 1|$. Evaluate $\int_{-3}^{1} |x + 1| dx$. What does this value represent on the graph?
14. Sketch the graph of $f(x) = \begin{cases} |x-2| + 2, & x \leq 2 \\ x^2 - 2, & x > 2 \end{cases}$

    Evaluate $\int_0^4 f(x)dx$. What does the value of this integral represent on the graph?

Answers:

1 .$\frac{128}{3}$ sq units. **2** .$\frac{9}{8}$ sq units. **3** .$\frac{21}{2}$ sq units. **4** .$\frac{1}{2}$ sq units.
5 .$\frac{1}{2}\left(\frac{\pi}{2} - 1\right) ab$ sq units. **6** .$\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq units **7**.$\left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right)$ sq units
8 .$\frac{15}{2}$ sq units **9** .7 sq units **10** .$\left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq units **12** .$\frac{9}{2}$ sq units

DIFFERENTIAL EQUATIONS

(Expected marks:8)

An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.

Order of a Differential Equation

The order of a differential equation is the order of the highest order derivative appearing in the equation.

Degree of a Differential Equation

The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.

Linear Differential Equations

A differential equation is a linear differential equation if it is expressible in the form $P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = Q$, where $P_0, P_1, P_2, \dots, P_{n-1}, P_n\ and\ Q$ are either constants or functions of independent variable x.

Formation of Differential Equations

To formulate a differential equation from a given relation containing independent variable and dependent variable and some arbitrary constants, we may follow the following procedure:

- *Write the given equation involving independent variable x, dependent variable y and the arbitrary constants.*
- *Obtain the number of arbitrary constants in the given equation. Let there be 'n' arbitrary constants.*
- *Differentiate the given equation* $\boldsymbol{n}$ *times with respect to* x
- *Eliminate arbitrary constants with the help of* $\boldsymbol{n}$ *equations involving differential coefficients obtained in previous step.*
  The equation so obtained is the desired differential equation.

Solution of a Differential Equation

The solution of a differential equation is a relation between the variables involved which satisfies the differential equation.

General Solution

The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation.

Particular Solution

Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.

- **Methods of solving a First Order First Degree Differential Equation**

➢ ***Equations in Variable Separable Form***

If the differential equation can be put in the form $f(x)dx = g(y)dy$ we say that the variables are separable and such equations can be solved by integrating on both sides. The solution is given by

$$\int f(x)\,dx = \int g(y)\,dy + c$$

where c is an arbitrary constant.

➢ **Homogeneous Differential Equations**

- A function $f(x, y)$ is called a homogeneous function of degree n, if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$
- If a first-order first degree differential equation is expressible in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree, then it is called a homogeneous differential equation.

Procedure to solve a Homogeneous Differential Equation

Step 1. Put the differential equation in the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

Step 2. Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the equation in step 1 and cancel out x from the right hand side. The equation reduces to the form

$$v + x\frac{dv}{dx} = F(v)$$

Step 3. Shift v on RHS and separate the variables v and x.

Step 4. Integrate both sides to obtain the solution in terms of v and x.

Step 5. Replace v by $\frac{y}{x}$ in the solution obtained in step 4 to obtain the solution in terms of x and y.

- *If the homogeneous differential equation is expressed as* $\frac{dx}{dy} = G\left(\frac{x}{y}\right)$ *we substitute* $x = vy$ *and* $\frac{dx}{dy} = v + y\frac{dv}{dy}$

➢ **<u>Linear Differential Equations</u>**

A differential equation is linear if the dependent variable (y) and its derivative appear only in first degree.

- ***The general form of a linear differential equation of first order is*** $\boldsymbol{\frac{dy}{dx} + Py = Q}$***, where P and Q are functions of x alone or constants.***

Procedure to solve a Linear Differential Equation

Step 1. Write the differential equation in the form $\frac{dy}{dx} + Py = Q$ and obtain P and Q.

Step 2. Find integrating factor $(I.F)$, given by $\boldsymbol{I.F = e^{\int P dx}}$

Step 3. Then the solution of the differential equation is

$$\boldsymbol{y(I.F) = \int Q(I.F)dx + c}$$

<u>NOTE:</u>

If the linear equation is of the form $\boldsymbol{\frac{dx}{dy} + Rx = S}$, where $\boldsymbol{R}$ and $\boldsymbol{S}$ are functions of y alone or constants, then $\boldsymbol{I.F = e^{\int R dy}}$

Then the solution of the differential equation is given by

$$\boldsymbol{x(I.F) = \int S(I.F)dy + c}$$

<u>Practice Questions</u>

1. Form the differential equation representing the family of curves $y = A\cos(x + B)$, where A and B are parameters.
2. Find the differential equation of all circles touches x −axis at the origin.

3. Form the differential equation of the family of ellipses having foci on x −axis and centre at the origin.
4. Show that the function $y = A\cos 2x - B\sin 2x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 4y = 0$
5. Solve $(1 + x^2)\frac{dy}{dx} - x = 2\tan^{-1} x$
6. Solve $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$
7. Solve $\frac{dy}{dx} = 1 + x + y + xy$
8. Solve $3e^x \tan y\, dx + (1 - e^x)sec^2 y dy = 0$
9. Solve $(x^2 + xy)dy = (x^2 + y^2)dx$
10. Solve $xdy - ydx = \sqrt{x^2 + y^2}dx$
11. Solve the differential equation $\frac{dy}{dx} - \frac{y}{x} = 2x^2 , x > 0$
12. Solve : $(1 + x^2)\frac{dy}{dx} - 2xy = (x^2 + 2)(x^2 + 1)$

HOTS

13. The normal lines to a given curve at each point pass through(2,0). The curve passes through(2,3). Formulate the differential equation and hence find out the equation of the curve.
14. The slope of the tangent of any point on a curve is λ times the slope of the line joining the point of contact to the origin. Formulate the differential equation and hence find the equation of the curve.
15. Solve $ydx - (x + 2y^2)dy = 0$
16. Solve the differential equation $(x^2 - y^2)dx + 2xy\, dy = 0$; given that $y = 1$ when $x = 1$.
17. Find the equation of the curve passing through the point $(-2,3)$given that the slope of the tangent to the curve at any point $(x, y)\ is\ \frac{2x}{y^2}$

Answers

1. $\frac{d^2y}{dx^2} + y = 0$, **2**. $(x^2 - y^2)\frac{dy}{dx} = 2xy$,
3. $xy\frac{d^2y}{d^2x} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$, **5**. $y = \frac{1}{2}\log|1 + x^2| + (\tan^{-1} x)^2 + c$
6. $y = 2\tan\frac{x}{2} - x + c$, **7**. $y = \pm e^{x+\frac{x^2}{2}+c} - 1$, **8**. $(e^x - 1)^3 = c\tan y$,
9. $c(x - y)^2 = |x|e^{\frac{-y}{x}}$, **10**. $\left(y + \sqrt{x^2 + y^2}\right)^2 = cx^4$, **11**. $y = x^3 + cx$,
12. $y = (x + \tan^{-1} x + c)(x^2 + 1)$, **13**. $y^2 = -(2 - x)^2 + 9$,
14. $y = cx^\lambda$, **15**. $x = 2y^2 + cy$, **16**. $x^2 + y^2 = 2x$,
17. $y^3 = 3x^2 + 15$